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LETTER TO THE EDITOR

Eight-vertex model and non-stationary Lamé equation

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Online at stacks.iop.org/JPhysA/38/L145**Abstract**

We study the ground state eigenvalues of Baxter's \mathbf{Q} -operator for the eight-vertex model in a special case when it describes the off-critical deformation of the $\Delta = -\frac{1}{2}$ six-vertex model. We show that these eigenvalues satisfy a non-stationary Schrödinger equation with the time-dependent potential given by the Weierstrass elliptic \wp -function where the modular parameter τ plays the role of (imaginary) time. In the scaling limit, the equation transforms into a 'non-stationary Mathieu equation' for the vacuum eigenvalues of the \mathbf{Q} -operators in the finite-volume massive sine-Gordon model at the super-symmetric point, which is closely related to the theory of dilute polymers on a cylinder and the Painlevé III equation.

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1. Introduction and summary

The \mathbf{Q} -operators introduced by Baxter in his pioneering paper [1] on the eight-vertex model continue to reveal their exceptional properties in the theory of integrable quantum systems. These operators play a central role in the remarkable connection of conformal field theory (CFT) with the spectral theory of the Schrödinger equation [2] discovered a few years ago [3]. As shown in [3, 4], the vacuum eigenvalues of the \mathbf{Q} -operators [5] in CFT, with the central charge $c < 1$, can be identified with the spectral determinants of certain one-dimensional stationary Schrödinger equations. Some further developments and applications of this connection can be found in the recent review article [6].

Apart from a few exceptions, the vacuum eigenvalues of the \mathbf{Q} -operators (considered as functions of the spectral parameter) do not generally satisfy any ordinary second-order differential equation themselves. One such exception is the case of the $c = 0$ CFT where these eigenvalues for particular Virasoro vacuum states are known to satisfy the Bessel differential equation [7]. Remarkably, a similar property holds for the lattice counterparts of these eigenvalues in the $\Delta = -\frac{1}{2}$ six-vertex model [8, 9] for the chain of an odd number of sites. In this letter, we explain how this property is generalized for the corresponding cases of the

lattice eight-vertex model and the massive finite volume sine-Gordon model with $N = 2$ supersymmetry.

We study the eight-vertex model on a periodic chain of an odd length, $N = 2n + 1$, $n = 0, 1, 2, \dots, \infty$. The eigenvalues of the transfer matrix of the model, $T(u)$, satisfy Baxter's famous TQ-equation

$$T(u)Q(u) = \phi(u - \eta)Q(u + 2\eta) + \phi(u + \eta)Q(u - 2\eta), \quad (1)$$

where u is the spectral parameter,

$$\phi(u) = \vartheta_1^N(u | q), \quad q = e^{i\pi\tau}, \quad \text{Im } \tau > 0, \quad (2)$$

and $\vartheta_1(u | q)$ is the standard theta function with the periods π and $\pi\tau$ (we follow the notation of [10]). Here we consider a special case $\eta = \pi/3$, where the ground state eigenvalue is known [11, 9] to have a very simple form for all (odd) N

$$T(u) = \phi(u), \quad \eta = \frac{\pi}{3}. \quad (3)$$

Equation (1) with this eigenvalue, $T(u)$, has two different solutions [5, 12], $Q_{\pm}(u) \equiv Q_{\pm}(u, q, n)$, which are entire functions of the variable u and obey the following periodicity conditions [1, 13]¹:

$$\begin{aligned} Q_{\pm}(u + \pi) &= \pm(-1)^n Q_{\pm}(u), & Q_{\pm}(u + \pi\tau) &= q^{-N/2} e^{-iNu} Q_{\mp}(u), \\ Q_{\pm}(-u) &= Q_{\pm}(u). \end{aligned} \quad (4)$$

The above requirements uniquely determine $Q_{\pm}(u)$ to within a common u -independent normalization factor. For further references, it is convenient to rewrite the functional equation for $Q_{\pm}(u)$ in the form

$$\phi(u)Q_{\pm}(u) + \phi\left(u + \frac{2\pi}{3}\right)Q_{\pm}\left(u + \frac{2\pi}{3}\right) + \phi\left(u + \frac{4\pi}{3}\right)Q_{\pm}\left(u + \frac{4\pi}{3}\right) = 0. \quad (5)$$

We show that the functions

$$\Psi_{\pm}(u) \equiv \Psi_{\pm}(u, q, n) = \frac{\vartheta_1^{2n+1}(u | q)}{\vartheta_1^n(3u | q^3)} Q_{\pm}(u, q, n), \quad (6)$$

which are meromorphic functions of the variable u for any fixed values of q and n , satisfy the non-stationary Schrödinger equation

$$6q \frac{\partial}{\partial q} \Psi(u, q, n) = \left\{ -\frac{\partial^2}{\partial u^2} + 9n(n+1)\wp(3u | q^3) + c(q, n) \right\} \Psi(u, q, n), \quad (7)$$

where the modular parameter τ plays the role of (imaginary) time and the time-dependent potential is defined through the elliptic Weierstrass \wp -function [10] (our function $\wp(v | e^{i\pi\epsilon})$ has the periods π and $\pi\epsilon$). The constant $c(q, n)$ appearing in (7) is totally controlled by the normalization of $Q_{\pm}(u)$ and can be explicitly determined once this normalization is fixed (see section 3).

Equation (7) is obviously related to the Lamé differential equation and could be naturally called the 'non-stationary Lamé equation'. To our knowledge this equation² (in fact, a more general equation) first explicitly appeared in [14] as a particular case of the Knizhnik–Zamolodchikov–Bernard equation [15, 16] for the one-point correlation function in the $sl(2)$ -WZW-model on the torus. Here, we will not explore this and other [17] interesting connections of equation (7) leaving that for the future.

¹ The factor $(-1)^n$ in (4) and (24) reflects our convention for labelling the eigenvalues for different n , which will be important in section 3.

² We are indebted to Professor I M Krichever for informing us about the work [14].

It is fairly trivial to show that the partial differential equation (7) for the meromorphic functions $\Psi_{\pm}(u)$ is equivalent to the Baxter equation (5). Indeed, every solution of (7), with the required analytic properties in the variable u , implied by (4) and (6), satisfies equation (5). However, the very existence of exactly two solutions with these properties is by no means trivial and reflects some rather special features of equation (7) discussed below.

Equation (7) has been discovered by virtue of a remarkable *polynomial property* of the eigenvalues $Q_{\pm}(u)$. Using a combination of analytical and numerical techniques, we have explicitly solved equation (5) for all values of $n \leq 10$. We have found that properly normalized eigenvalues $Q_{\pm}(u)$ could always be written as

$$Q_{\pm}(u, q, n) = P_{2n+1}^{(\pm)} \left(\vartheta_3 \left(\frac{u}{2} \middle| q^{1/2} \right), \vartheta_4 \left(\frac{u}{2} \middle| q^{1/2} \right), \gamma \right), \quad \gamma = - \left[\frac{\vartheta_1(\frac{\pi}{3} | q^{1/2})}{\vartheta_2(\frac{\pi}{3} | q^{1/2})} \right]^2, \quad (8)$$

where $P_{2n+1}^{(\pm)}(\alpha, \beta, \gamma)$ are homogeneous polynomials of the degree $2n + 1$ in the variables α and β , with coefficients being polynomials in the variable γ with *integer coefficients*. Then we considered a class of linear second-order partial differential equations in two variables u and q , where the coefficients of the second-order derivatives are independent of n and all other coefficients are at most second degree polynomials in n . Equation (7) was then found as the only equation in this class satisfied by (6) with all explicitly calculated polynomials (8) with $n \leq 10$. It turned out that this equation uniquely defines two and only two such polynomials (8) for every value $n = 0, 1, 2, \dots, \infty$. It would be interesting to clarify the combinatorial nature of these polynomials, given that the related $\Delta = -\frac{1}{2}$ six-vertex model is connected to various important enumeration problems [18, 19].

In the scaling limit

$$n \rightarrow \infty, \quad q \rightarrow 0, \quad t = 8nq^{3/2} = \text{fixed}, \quad (9)$$

the functions (6) essentially reduce to the ground state eigenvalues $Q_{\pm}(\theta) \equiv Q_{\pm}(\theta, t)$ of the \mathbf{Q} -operators of the restricted massive sine-Gordon model (at the so-called super-symmetric point) on a cylinder of the spatial circumference R , where $t = MR$ and M is the soliton mass. Equations (1) and (4) become

$$Q_{\pm}(\theta) = Q_{\pm}(\theta + 2\pi i) + Q_{\pm}(\theta - 2\pi i), \quad (10)$$

$$Q_{\pm}(\theta + 3\pi i) = e^{\pm \frac{i\pi}{2}} Q_{\pm}(\theta), \quad Q_+(\theta) = Q_-(-\theta), \quad (11)$$

where the variable θ is defined as $u = \pi \tau / 2 - i\theta / 3$. With a suitable t -dependent normalization of $Q_{\pm}(\theta)$, equation (7) could be brought to a particularly simple form

$$t \frac{\partial}{\partial t} Q_{\pm}(\theta, t) = \left\{ \frac{\partial^2}{\partial \theta^2} - \frac{1}{8} t^2 (\cosh 2\theta - 1) \right\} Q_{\pm}(\theta, t). \quad (12)$$

With the same normalization the asymptotic behaviour of $Q_{\pm}(\theta)$ at large θ is given by

$$\log Q_{\pm}(\theta) = -\frac{1}{4} t e^{\theta} + \log \mathcal{D}_{\pm}(t) + 2(\partial_t \log \mathcal{D}_{\pm}(t) - t/8) e^{-\theta} + 2(\partial_t^2 \log \mathcal{D}_{\pm}(t) - \partial_t \log \mathcal{D}_{\pm}(t)/t) e^{-2\theta} + O(e^{-3\theta}), \quad \theta \rightarrow +\infty, \quad (13)$$

where $\mathcal{D}_{\pm}(t)$ are the Fredholm determinants which previously appeared in connection with the calculation of the ‘supersymmetric index’ and the problem of dilute polymers on a cylinder [20–23]. Note, in particular, that the quantity

$$F(t) = \frac{d}{dt} U(t), \quad U(t) = \log \frac{\mathcal{D}_+(t)}{\mathcal{D}_-(t)}, \quad (14)$$

describes the free energy of a single incontractible polymer loop and satisfies the Painlevé III

equation [20]

$$\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} U(t) = \frac{1}{2} \sinh 2U(t). \quad (15)$$

In this connection, it is useful to mention the other celebrated appearances of Painlevé transcendents in the theory of the two-dimensional Ising model [24] and in the problem of isomonodromic transformations of the second-order differential equations [25].

We did not attempt to make this letter self-contained. Detailed results are presented in [26]. The main reference for the eight-vertex model and its commuting \mathbf{T} - and \mathbf{Q} -matrices is Baxter's original paper [1]; section 2 is meant to be read in conjunction with this paper. The definitions of the \mathbf{T} - and \mathbf{Q} -operators in continuous quantum field theory are given in [5, 27]. The connection of the \mathbf{Q} -operators with the problem of dilute polymers is explained in [7].

2. The eight-vertex model and TQ-relation

We consider the eight-vertex model on the N -column square lattice with periodic boundary conditions and assume that N is an odd integer $N = 2n + 1$. Following [1], we parametrize the Boltzmann weights a, b, c, d of the eight-vertex model as³

$$\begin{aligned} a &= \rho \vartheta_4(2\eta | q^2) \vartheta_4(u - \eta | q^2) \vartheta_1(u + \eta | q^2), \\ b &= \rho \vartheta_4(2\eta | q^2) \vartheta_1(u - \eta | q^2) \vartheta_4(u + \eta | q^2), \\ c &= \rho \vartheta_1(2\eta | q^2) \vartheta_4(u - \eta | q^2) \vartheta_4(u + \eta | q^2), \\ d &= \rho \vartheta_1(2\eta | q^2) \vartheta_1(u - \eta | q^2) \vartheta_1(u + \eta | q^2), \end{aligned} \quad (16)$$

and fix the normalization factor ρ as

$$\rho = 2\vartheta_2(0 | q)^{-1} \vartheta_4(0 | q^2)^{-1}. \quad (17)$$

It is convenient to introduce new variables γ and x , where γ is defined by (cf [1]),

$$\gamma = \frac{(a - b + c - d)(a - b - c + d)}{(a + b + c + d)(a + b - c - d)} = - \left[\frac{\vartheta_1(\eta | q^{1/2})}{\vartheta_2(\eta | q^{1/2})} \right]^2, \quad (18)$$

and x satisfies the quadratic equation

$$\left(\sqrt{x} - \frac{\gamma}{\sqrt{x}} \right)^2 = - \frac{16(a - b)^2 cd}{(c + d)^2 (a + b + c + d)(a + b - c - d)}, \quad (19)$$

where we choose the following root:

$$x = \gamma \frac{\overline{\vartheta}_3^2(u)}{\overline{\vartheta}_4^2(u)}, \quad \overline{\vartheta}_3(u) = \vartheta_3\left(\frac{u}{2} \middle| q^{1/2}\right), \quad \overline{\vartheta}_4(u) = \vartheta_4\left(\frac{u}{2} \middle| q^{1/2}\right). \quad (20)$$

The row-to-row transfer-matrices $\mathbf{T}(u)$ and the $\mathbf{Q}(u)$ -matrices of the model form a commutative family; their eigenvalues satisfy the functional relation (1). In [1] Baxter explicitly constructed the matrix $\mathbf{Q}(u)$ provided that $\eta = \frac{\pi m}{2L}$ for integers m and L . We only consider a special case when the weights (16) satisfy the constraint [9]

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab), \quad (21)$$

which is equivalent to the condition $\eta = \pi/3$. In [9, 28], it was conjectured that the largest eigenvalue of the transfer matrix (corresponding to the double-degenerate ground state of the

³ We use the notation of [10] for theta-functions $\vartheta_k(u | q)$, $k = 1, 2, 3, 4$, of the periods π and $\pi\tau$, $q = e^{i\pi\tau}$, $\text{Im } \tau > 0$. The theta-functions $\mathbf{H}(v)$, $\Theta(v)$ of the nome q_B used in [1] are given by

$$q_B = q^2, \quad \mathbf{H}(v) = \vartheta_1\left(\frac{\pi v}{2K_B} \middle| q^2\right), \quad \Theta(v) = \vartheta_4\left(\frac{\pi v}{2K_B} \middle| q^2\right),$$

where K_B is the complete elliptic integral of the first kind with the nome q_B .

model) has the simple form, (3),

$$T(u) = (a + b)^N = \phi(u). \tag{22}$$

Here we study the corresponding eigenvalues $Q_{\pm}(u)$ of the \mathbf{Q} -matrix. As noted in [29], the method used in [1] for the construction of the \mathbf{Q} -matrix cannot be executed in its full strength for $\eta = \pi/3$, since some axillary \mathbf{Q} -matrix, $\mathbf{Q}_R(u)$, in [1] is not invertible in the full 2^N -dimensional space of states of the model. The numerical results presented in table 1 of [29] make it natural to suggest that the rank of $\mathbf{Q}_R(u)$ in this case is given by the N th Lucas number $((1 + \sqrt{5})/2)^N + ((1 - \sqrt{5})/2)^N$ which coincides with the dimension of the space of states of the N -site hard hexagon model [30]⁴. This indicates that for $\eta = \pi/3$ the construction of [1] only provides a ‘restricted’ $\mathbf{Q}^{(r)}(u)$ -matrix which acts in some ‘RSOS-projected’ subspace of the full space of states of the eight-vertex model. This interesting phenomenon certainly deserves special investigations in its own right; we have verified for several small values of N that the ground state eigenvectors corresponding to (22) belong to this RSOS-projected subspace and that the eigenvalues of $\mathbf{Q}^{(r)}(u)$ satisfy (1). Below we will assume that this is true for all (odd) N .

After this brief review, let us now describe our main results. For $\eta = \pi/3$ the variable $\gamma \equiv \gamma(q)$, defined by (18), depends on q only, while the variable $x \equiv x(u, q)$, defined by (19), depends on u and q . Below it will be more convenient to use the combinations

$$Q_1(u) = (Q_+(u) + Q_-(u))/2, \quad Q_2(u) = (Q_+(u) - Q_-(u))/2, \tag{23}$$

which are simply related by the periodicity relation

$$Q_{1,2}(u + \pi) = (-1)^n Q_{2,1}(u). \tag{24}$$

Bearing in mind this simple relation we will only quote results for $Q_1(u)$, writing it as $Q_1^{(n)}(u)$ to indicate the n -dependence. We have found that all the eigenvalues $Q_1^{(n)}(u)$ can be written as

$$Q_1^{(n)}(u) = \mathcal{N}(q, n) \bar{\vartheta}_3(u) \bar{\vartheta}_4^{2n}(u) \mathcal{P}_n(x, z), \quad z = \gamma^{-2}, \tag{25}$$

where $\mathcal{N}(q, n)$ is an arbitrary normalization factor and $\mathcal{P}_n(x, z)$ are polynomials in x, z of the degree n in x ,

$$\mathcal{P}_n(x, z) = \sum_{k=0}^n r_k^{(n)}(z) x^k, \tag{26}$$

while $r_i^{(n)}(z), i = 0, \dots, n$, are polynomials in z with integer coefficients. The normalization of $\mathcal{P}_n(x, z)$ is fixed by the requirement $r_n^{(n)}(0) = 1$. The polynomials $\mathcal{P}_n(x, z)$ are uniquely determined by the following partial differential equation in the variables x and z :

$$\{A(x, z)\partial_x^2 + B_n(x, z)\partial_x + C_n(x, z) + T(x, z)\partial_z\} \mathcal{P}_n(x, z) = 0, \tag{27}$$

where

$$A(x, z) = 2x(1 + x - 3xz + x^2z)(x + 4z - 6xz - 3xz^2 + 4x^2z^2), \tag{28}$$

$$B_n(x, z) = 4(1 + x - 3xz + x^2z)(x + 3z - 7xz + 3x^2z^2) + 2nx(1 - 14z + 21z^2 - 8x^3z^3 + 3x^2z(3z^2 + 6z - 1) - x(1 - 9z + 23z^2 + 9z^3)) \tag{29}$$

$$C_n(x, z) = n[z(9z - 5) + x^2z(3z^2 + 11z - 2) + x(9z^3 - 38z^2 + 19z - 2) - 4x^3z^3 + nz(1 - 9z - x(9z^2 - 36z + 3) + x^2(3z^2 - 31z + 4) + 8x^3z^2)] \tag{30}$$

$$T(x, z) = -2z(1 - z)(1 - 9z)(1 + x - 3xz + x^2z). \tag{31}$$

⁴ Sergeev noted [31] that only a minor modification of the arguments of [1] leads to the matrix \mathbf{Q}_R with the rank equal to 2^N for even N and to $(2^N - 2)$ for odd N .

The first few polynomials $\mathcal{P}_n(x, z)$ read

$$\mathcal{P}_0(x, z) = 1, \quad \mathcal{P}_1(x, z) = x + 3, \quad \mathcal{P}_2(x, z) = x^2(1 + z) + 5x(1 + 3z) + 10, \quad (32)$$

$$\mathcal{P}_3(x, z) = x^3(1 + 3z + 4z^2) + 7x^2(1 + 5z + 18z^2) + 7x(3 + 19z + 18z^2) + 35 + 21z. \quad (33)$$

In the next section, we will prove that the eigenvalues $Q_1^{(n)}(u)$ given by (25) (as well as all related eigenvalues $Q_2^{(n)}(u)$ and $Q_{\pm}^{(n)}(u)$ given by (23) and (24) automatically satisfy the functional relation (5), merely as a consequence of the defining property (27) of $\mathcal{P}_n(x, z)$. Of course, for small values of n this functional relation can be checked directly. For example, it is not very difficult to check it for

$$Q_1^{(0)}(u) \sim \bar{\vartheta}_3(u), \quad Q_1^{(1)}(u) \sim \bar{\vartheta}_3(u)[\gamma \bar{\vartheta}_3^2(u) + 3\bar{\theta}_4^2(u)], \quad (34)$$

by employing various identities for elliptic functions; however for $n \geq 2$ this does not appear to be practical.

The polynomials $\mathcal{P}_n(x, z)$ can be effectively calculated with the following procedure. It is easy to see that (27) leads to descending recurrence relations for coefficients in (26) in the sense that each coefficient $r_k^{(n)}(z)$ with $k < n$ can be recursively calculated in terms of $r_m^{(n)}(z)$, with $m = k + 1, \dots, n$ and, therefore, can be eventually expressed through the coefficient $s_n(z) \equiv r_n^{(n)}(z)$ of the leading power of x . These leading coefficients, $s_n(z)$, $n = 0, 1, 2, \dots, \infty$, are uniquely determined by the following recurrence relation:

$$2z(z-1)(9z-1)^2[\log s_n(z)]'_z + 2(3z-1)^2(9z-1)[\log s_n(z)]'_z + 8(2n+1)^2 \frac{s_{n+1}(z)s_{n-1}(z)}{s_n^2(z)} - [4(3n+1)(3n+2) + (9z-1)n(5n+3)] = 0, \quad (35)$$

with the initial condition $s_0(z) = s_1(z) \equiv 1$. In particular, for $z = 1/9$ (corresponding to $q = 0$) this gives

$$s_0\left(\frac{1}{9}\right) = 1, \quad s_{n+1}\left(\frac{1}{9}\right) = \frac{2^{n-1}n!(3n+2)!}{3^n((2n+1)!)^2} s_n\left(\frac{1}{9}\right). \quad (36)$$

Currently, the polynomials $\mathcal{P}_n(x, z)$ have been calculated explicitly for $n \leq 50$; using the above procedure they can be easily calculated for higher values of n .

3. Non-stationary Lamé equation

Evidently, the algebraic form of the partial differential equation given by (27) is not very illuminating (even though it is quite useful for the analysis of polynomial solutions). Fortunately, this equation has a much more elegant equivalent form given by the non-stationary Lamé equation (7) discussed in the introduction. The details of transformations between these two forms will be presented elsewhere [26]. Noting the frustrating expressions for the coefficients in (27), it is not surprising that these transformations turn out to be rather tedious. They involve many elliptic function identities, in particular, the algebraic properties of the ring of theta constants and their q -derivatives [32] happened to be extremely useful.

Let us choose the normalization factor $\mathcal{N}(q, n)$ in (25) as

$$(-1)^n 2^{(3n^2+5n+1)/2} s_n\left(\frac{1}{9}\right) \mathcal{N}(q, n) = i^n \gamma^{-n} \vartheta_1'(0|q)^{-n-1/3} \{(\gamma^2 - 1) \vartheta_4^4(0|q)\}^{\frac{n(n+1)}{2}} \quad (37)$$

and define the functions $\Psi_{\pm}(u, q, n)$ as in (6) where $Q_{\pm}(u)$ are given by (23), (24) and (25). Then equation (27) takes the form (7) where the constant term $c(q, n)$ is given by

$$c(q, n) = -3n(n+1) \frac{\vartheta_1'''(0|q^3)}{\vartheta_1'(0|q^3)}. \quad (38)$$

By construction the functions $\Psi_{\pm}(u, q, n)$ are meromorphic functions of the variable u , which obey periodicity relations

$$\Psi(u + 2\pi) = \Psi(u), \quad \Psi(u + 2\pi\tau) = q^{-6} e^{-6iu} \Psi(u), \quad \Psi(-u) = (-1)^{n+1} \Psi(u) \tag{39}$$

and have $(n + 1)$ th-order zeros at $u = k\pi + m\pi\tau$, where $k, m \in \mathbb{Z}$, i.e.,

$$\Psi(\varepsilon + k\pi + m\pi\tau) = O(\varepsilon^{n+1}), \quad \varepsilon \rightarrow 0, \quad k, m \in \mathbb{Z}. \tag{40}$$

Let us show that for $n \geq 1$, equation (7), restricted to a class of functions $\Psi(u)$ with such analytic properties, is equivalent to the functional relation (5). Any such solution of (7), $\Psi(u)$, could have either an $(n + 1)$ th-order zero or an n th-order pole in the variable u , at all points $u = (3k \pm 1)\pi/3 + m\pi\tau$, with $k, m \in \mathbb{Z}$, and these are the only points where $\Psi(u)$ could have poles. Consider the function

$$\Phi(u) = \Psi(u) + \Psi\left(u + \frac{2\pi}{3}\right) + \Psi\left(u + \frac{4\pi}{3}\right), \tag{41}$$

which also satisfies (7) along with $\Psi(u)$. When $u = 0$ the second and third terms in (41) may have n th-order poles; however, they must cancel each other due to the last relation in (39). Thus, $\Phi(u)$ could have at most $(n - 1)$ th-order pole at $u = 0$ (the first term in (41) obviously vanishes due to (40)). As noted above, such a pole is forbidden by equation (7) and thus $\Phi(u)$ has the $(n + 1)$ th order zero at $u = 0$. Similarly, one concludes that $\Phi(u)$ vanishes at all points $u = k\pi/3 + m\pi\tau$, with $k, m \in \mathbb{Z}$, and, therefore, has at least $12(n + 1)$ zeros in the periodicity parallelogram (of the periods 2π and $2\pi\tau$) and no poles at all. However, for $n \geq 1$ this contradicts (39), unless $\Phi(u) \equiv 0$, which is equivalent to (5). The special case $n = 0$ is considered in [26].

Now consider various limiting forms of equation (7). When $q \rightarrow 0$, with u and n kept fixed, the functions $\Psi_{\pm}(u)$ reduce to their analogues for the six-vertex model

$$\Psi_{\pm}(u, q, n) = q^{\frac{3}{2}(d_{\pm} + \frac{1}{4})} \Psi_{\pm}^{(6v)}(u, n)(1 + O(q)), \quad q \rightarrow 0, \tag{42}$$

where $d_{\pm} = (1 \mp 6)/36$ and equation (7) becomes

$$\left\{ -\frac{d^2}{ds^2} + \frac{n(n+1)}{\sin^2 s} - \left(d_{\pm} + \frac{1}{4} \right) \right\} \Psi_{\pm}^{(6v)}(s/3, n) = 0, \tag{43}$$

which is simply related to equation (13) of [9]. Taking now $n \rightarrow \infty$ and $s \sim i \log n$ one recovers the eigenvalues of the \mathbf{Q} -operators of [5] corresponding to $p = \pm 1/6$ vacuum states in the $c = 0$ CFT

$$\mathcal{Q}_{\pm}^{(\text{CFT})}(\theta + \log t) = e^{\frac{\theta}{3}} \lim_{n \rightarrow \infty} \Psi_{\pm}^{(6v)}\left(\frac{i}{3} \log(8n/t) - \frac{i}{3}\theta\right), \tag{44}$$

which are known [7] to satisfy the Bessel differential equation

$$\left\{ -\partial_{\theta}^2 + \partial_{\theta} + \frac{1}{16}t^2 e^{2\theta} + d_{\pm} \right\} \mathcal{Q}_{\pm}^{(\text{CFT})}(\theta + \log t) = 0. \tag{45}$$

In the most interesting scaling limit (9), the limiting functions

$$\mathcal{Q}_{\pm}(\theta, t) = t^{1/4} e^{t^2/16} e^{\theta/2} \lim_{n \rightarrow \infty} n^{-1/4} \Psi_{\pm}(\pi\tau/2 - i\theta/3, e^{i\pi\tau}, n) \Big|_{\tau = \frac{2i}{3\pi} \log(8n/t)} \tag{46}$$

coincide with the eigenvalues [7] of the \mathbf{Q} -operators for special twisted vacuum states in the massive sine-Gordon at the supersymmetric point [22] (where the ground state energy of the model vanishes identically due to the supersymmetry). These eigenvalues satisfy the ‘non-stationary Mathieu equation’ (12). Let us show that this equation completely determines the asymptotic expansion of these eigenvalues at large θ ,

$$\log \mathcal{Q}_{\pm}(\theta, t) = -\frac{t}{4} e^{\theta} + \sum_{k=0}^{\infty} \mathcal{B}_{\pm}^{(k)}(t) e^{-k\theta}, \quad \theta \rightarrow +\infty. \tag{47}$$

Consider the integral operator $\hat{\mathcal{K}}(t)$ with the kernel

$$\mathcal{K}(t | \theta, \theta') = \frac{1}{2\pi} \frac{e^{-u(\theta)-u(\theta')}}{1 + e^{\theta-\theta'}}, \quad u(\theta) = \frac{t}{2} \cosh \theta, \quad (48)$$

which satisfies the following identity:

$$[-t\partial_t + \hat{\mathcal{M}}_\theta - \hat{\mathcal{M}}_{\theta'}]\mathcal{K}(t | \theta, \theta') = \frac{1}{4\pi} t e^{-u(\theta)-u(\theta')+\theta'}, \quad (49)$$

where $\hat{\mathcal{M}}_\theta$ denotes the differential operator in the RHS of (12). Using this identity, one can show that $\mathcal{Q}_\pm(\theta)$ satisfy the linear integral equation discovered in [7]

$$\mathcal{Q}_\pm(\theta, t) = \mathcal{D}_\pm(t) e^{-u(\theta)} \mp \int_{-\infty}^{\infty} \mathcal{K}(t | \theta, \theta') \mathcal{Q}_\pm(\theta', t) d\theta', \quad (50)$$

provided that the functions $\mathcal{D}_\pm(t)$ coincide with the Fredholm determinants [22]

$$\mathcal{D}_\pm(t) = \mathcal{C}_\pm \det(1 \pm \hat{\mathcal{K}}(t)), \quad (51)$$

where \mathcal{C}_\pm are numerical constants. Comparing (47) with (50) one concludes that $\mathcal{B}_\pm^{(0)}(t) = \log \mathcal{D}_\pm(t)$. Then equation (12) allows us to express all higher coefficients in (47) through the power $\mathcal{B}_\pm^{(0)}(t)$ and its derivatives. The few first coefficients are shown in (13). Finally note that in the limit $t \rightarrow 0$, $\theta \sim -\log t$,

$$\mathcal{Q}_\pm(\theta, t) = t^{d_\pm} \mathcal{Q}_\pm^{(\text{CFT})}(\theta + \log t)(1 + O(t^{4/3})), \quad t \rightarrow 0, \quad \theta \sim -\log t, \quad (52)$$

and equation (12) reduces to (45) as it, of course, should⁵.

We expect that our results can be readily extended to elliptic generalizations of the other special lattice models [33], closely related to the $\eta = \pi/3$ six-vertex model. However, it would be more important to understand whether similar considerations could be applied for the eight-vertex with an arbitrary value of η and whether the general scheme of correspondence between the $c < 1$ CFT and ordinary differential equations, developed in [3, 4, 34], can be extended to the massive field theory (sine-Gordon model) with the use of partial differential equations. The investigation of these questions is in progress.

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⁵ The normalization of integral operator (48) has been fixed from the comparison of the $t \rightarrow 0$ limit of the coefficients $\mathcal{B}_\pm^{(1)}(t)$ with appropriate CFT results of [5]. This normalization corresponds to $\lambda = 1/(4\pi)$ in equation (3.1) of [22].

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